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Spontaneous magnetization of the superintegrable chiral Potts model: calculation of the determinant D_{PQ}

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Abstract

For the Ising model, the calculation of the spontaneous magnetization leads to the problem of evaluating a determinant. Yang did this by calculating the eigenvalues in the large-lattice limit. Montroll, Potts and Ward expressed it as a Toeplitz determinant and used Szegő's theorem: this is almost certainly the route originally travelled by Onsager. For the corresponding problem in the superintegrable chiral Potts model, neither approach appears to work: here we show that the determinant D_{PQ} can be expressed as that of a product of two Cauchy-like matrices. One can then use the elementary exact formula for the Cauchy determinant. One of course regains the known result, originally conjectured in 1989.

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1. Introduction

Extrapolating from the Ising case and from series expansions, Albertini *et al* conjectured in 1989 [1] that the order parameter or spontaneous magnetization of the solvable N -state chiral Potts model is

$$\mathcal{M}_r = (1 - k'^2)^{r(N-r)/2N^2}. \quad (1.1)$$

Here, r is an integer, between 0 and N , and k' is a parameter that is 'universal' in the sense that it is the same for all rows and columns of the lattice, even for an inhomogeneous model where the rapidities vary from row to row and from column to column [2]. Here we consider the ferromagnetically ordered phase of the system, where $0 < k' < 1$. This k' is small at low temperatures (high order), and tends to 1 at the critical temperature (vanishing order), so we can regard it as a 'temperature'.

The author was able to derive result (1.1) in 2005 [3, 4] using an analytic method based on functional relations satisfied by generalized order parameters in the large-lattice limit.

If the vertical rapidities of the homogeneous model take a particular value, or if those of a model with alternating vertical rapidities satisfy a certain relation [5, 6], then we obtain the

‘superintegrable’ case of the chiral Potts model. For this case, von Gehlen and Rittenberg [7] showed that the horizontal and vertical components of the transfer matrix satisfy the ‘Dolan–Grady’ condition [8], which ensures that they generate the Onsager algebra. This is the algebra generated by the transfer matrices of the Ising model [9, equations (59)–(61)], [10, equation (4.12)], [11–18]. Onsager used it to calculate the free energy of the Ising model.

In fact the superintegrable chiral Potts model looks very much like the Ising model. (For $N = 2$ it is the Ising model.) If one considers the model on a cylinder of L columns, with the spins on the top and bottom rows fixed to the value zero, then the row-to-row transfer matrices can be reduced from dimension N^L to dimension 2^m , where m is not greater than L . Like the Ising model, the partition function is a matrix element of a direct product of m matrices, each of dimension 2.

For the Ising model, one can calculate the correlations quite explicitly as determinants by using free-fermion operators [19, 20], or equivalently by writing the partition functions directly as pfaffians [21].

If the partition functions of the N -state superintegrable model resemble those of the Ising model, then perhaps the correlations are also similar, and can be obtained by similar methods. In particular, the order parameter can be defined as

$$\mathcal{M}_r = \langle \omega^{ra} \rangle, \tag{1.2}$$

where $r = 1, 2, \dots, N - 1$,

$$\omega = e^{2\pi i/N} \tag{1.3}$$

and a is a particular spin inside the lattice. We can use this definition for a finite lattice: we do so herein. We only expect the simple result (1.1) to be true in the limit when the lattice is large and the spin a is deep inside it. It should then be independent of the values of the rapidity parameters, so should be the same for the general solvable model as for the superintegrable case.

We take the spins to have the values $0, \dots, N - 1$ and $\sigma = \{\sigma_1, \dots, \sigma_L\}$ to be the set of spins in a horizontal row of the lattice. Let u_b be the N^L -dimensional vector with entries

$$(u_b)_\sigma = \delta(\sigma_1, b)\delta(\sigma_2, b) \cdots \delta(\sigma_L, b) \tag{1.4}$$

and S_r be the diagonal matrix with entries

$$(S_r)_{\sigma,\sigma'} = \omega^{r\sigma_1} \prod_{j=1}^L \delta(\sigma_j, \sigma'_j). \tag{1.5}$$

Then with these boundary conditions, (1.2) can be written in terms of the row-to-row transfer matrix T as

$$\mathcal{M}_r = \frac{u_0^\dagger T^m S_r T^n u_0}{u_0^\dagger T^{m+n} u_0}, \tag{1.6}$$

where m (n) is the number of rows below (above) the particular spin a . We have chosen a to lie in the first column: since we are using cylindrical (cyclic) boundary conditions, this is no restriction.

The transfer matrix T commutes with a Hamiltonian \mathcal{H} . For convenience, we replace (1.6) with

$$\mathcal{M}_r^{(1)} = \frac{u_0^\dagger e^{-\alpha\mathcal{H}} S_r e^{-\beta\mathcal{H}} u_0}{u_0^\dagger e^{-(\alpha+\beta)\mathcal{H}} u_0}. \tag{1.7}$$

In the limit of m, n large, the only eigenvectors of T entering the RHS of (1.6) are those corresponding to the N asymptotically degenerate largest eigenvalues. The same is true of

(1.7) in the limit of α, β large and positive. Hence these limits of (1.6), (1.7) must be the same.

Let R be the operator that increases every spin in a row by 1: its elements are

$$R_{\sigma, \sigma'} = \prod_{j=1}^L \delta(\sigma_j, \sigma'_j + 1). \quad (1.8)$$

Since $R^N = 1$, its eigenvalues are $1, \omega, \omega^2, \dots, \omega^{N-1}$. Let \mathcal{V}_P (for $P = 0, 1, \dots, N - 1$) be the space of vectors v such that

$$Rv = \omega^P v. \quad (1.9)$$

Then the full N^L -dimensional space is the union of $\mathcal{V}_0, \dots, \mathcal{V}_{N-1}$. Let v_P be the vector

$$v_P = N^{-1/2} \sum_{b=0}^{N-1} \omega^{-Pb} u_b. \quad (1.10)$$

Then v_P and $e^{-\beta\mathcal{H}}v_P$ are in \mathcal{V}_P . If

$$Q = P + r, \quad \text{mod } N, \quad (1.11)$$

then $\mathcal{S}_r e^{-\beta\mathcal{H}}v_Q$ is also in \mathcal{V}_P . Vectors in different spaces $\mathcal{V}_P, \mathcal{V}_Q$ are orthogonal. It follows that, for $b = 0, \dots, N - 1$,

$$u_b^\dagger e^{-\alpha\mathcal{H}} \mathcal{S}_r e^{-\beta\mathcal{H}} u_0 = N^{-1} \sum_{P=0}^{N-1} \omega^{-Pb} W_{P, P+r}(\alpha, \beta) \quad (1.12a)$$

$$u_b^\dagger e^{-\alpha\mathcal{H}} u_0 = N^{-1} \sum_{P=0}^{N-1} \omega^{-Pb} Z_P(\alpha), \quad (1.12b)$$

where

$$W_{PQ}(\alpha, \beta) = v_P^\dagger e^{-\alpha\mathcal{H}} \mathcal{S}_r e^{-\beta\mathcal{H}} v_Q, \quad (1.13a)$$

$$Z_P(\alpha) = v_P^\dagger e^{-\alpha\mathcal{H}} v_P. \quad (1.13b)$$

Each LHS of (1.12) is the partition function of a lattice where the top spins are fixed to the value zero, and the bottom to the value b . If $b \neq 0$, this ensures that there is at least one mismatched seam (between phases where most spins are zero and most spins have the value b) running horizontally across the lattice. If ζ is the interfacial tension, (which we expect to be independent of L), then this will make each partition function smaller than that for $b = 0$ by a factor $e^{-L\zeta}$ [22, section 7.10].

In the limit of L large the ratio of these expressions for $b \neq 0$ to their values for $b = 0$ will therefore become zero. From (1.12), it follows that in this limit $Z_P(\alpha)$ is independent of P , while $W_{PQ}(\alpha, \beta)$ depends on P, Q only via their difference $r = Q - P$. We show at the end of section 5 that these assertions are certainly true in the limit when α, β, L all tend to infinity.

The numerator in (1.7) can therefore be replaced by $W_{PQ}(\alpha, \beta)$, for any P, Q satisfying (1.11). The denominator can be replaced by $Z_P(\alpha + \beta)$, for any P , but if α is also large, each $Z_P(\alpha)$ is of the form $K e^{v\alpha}$, where K, v must be independent of P , so we can more symmetrically replace the denominator by $[Z_P(2\alpha)Z_Q(2\beta)]^{1/2}$, giving as our final expression for the spontaneous magnetization

$$\mathcal{M}_r^{(2)} = \frac{W_{PQ}(\alpha, \beta)}{[Z_P(2\alpha)Z_Q(2\beta)]^{1/2}}. \quad (1.14)$$

The three expressions $\mathcal{M}_r, \mathcal{M}_r^{(1)}, \mathcal{M}_r^{(2)}$ are equal in the limit L, m, n or L, α, β all becoming infinite.

Previous calculations for finite L, α, β

In [23–25], we have looked at the problem of calculating $W_{PQ}(\alpha, \beta)$, $Z_P(\alpha)$ and hence $\mathcal{M}_r^{(1)}$, $\mathcal{M}_r^{(2)}$, algebraically, for the superintegrable chiral Potts model, with finite L, α, β . We refer to these papers as I, II, III, and prefix their equations accordingly.

The calculation of $Z_P(\alpha)$ is straightforward, being a minor adaptation of the partition function calculations of [5, 6], and is given in paper II. The real problem is to calculate $W_{PQ}(\alpha, \beta)$, or equivalently the ratio

$$\mathcal{D}_{PQ}(\alpha, \beta) = \frac{W_{PQ}(\alpha, \beta)}{Z_P(\alpha) Z_Q(\beta)}. \tag{1.15}$$

If we also define

$$\mathcal{Z}_P(\alpha) = Z_P(2\alpha)/Z_P(\alpha)^2, \tag{1.16}$$

then (1.14) becomes

$$\mathcal{M}_r^{(2)} = \frac{\mathcal{D}_{PQ}(\alpha, \beta)}{[\mathcal{Z}_P(\alpha)\mathcal{Z}_Q(\beta)]^{1/2}}. \tag{1.17}$$

In I we considered the $N = 2$ case, which is the Ising model. We used Kaufman’s spinor operators (Clifford algebra) [26] to first write $\mathcal{D}_{PQ}(\alpha, \beta)$ (for $P = 0, Q = r = 1$) as the square root of an L by L determinant in I.4.59. Then in section 6 of I, equations (I.6.29) and (I.7.9), we further reduced this result to an m by m determinant (with no square root), where $m \leq L/2$. With obvious modifications of notation to allow for the working of the later papers, and taking $\rho = 0, x = 1$ in (I.3.5), (I.3.6) (also in (II.5.23) and (II.5.25)), this result can be written as

$$\mathcal{D}_{PQ}(\alpha, \beta) = \det[I_m - X_P(\alpha)E_{PQ}B_{PQ}X_Q(\beta)E_{QP}B_{QP}], \tag{1.18}$$

where I_m is the identity matrix of dimension m , $X_P(\alpha), E_{PQ}, X_Q(\beta), E_{QP}$ are diagonal matrices, B_{PQ} is an m by m' matrix, where $|m - m'| = 0$ or 1 , $B_{QP} = -B_{PQ}^T$ and B_{PQ} is orthogonal in the sense that

$$B_{PQ}^T B_{PQ} = I_{m'} \quad \text{if } m \geq m', \quad B_{PQ} B_{PQ}^T = I_m \quad \text{if } m \leq m'. \tag{1.19}$$

We used the first (L by L) form to take the limit $\alpha, \beta \rightarrow +\infty$: the result of course agreed with that of Yang [20] and Montroll, Potts and Ward [21] and with (1.1) above.

In II we considered the superintegrable chiral Potts model and showed that the N^L -dimensional matrices in (1.13a) could be replaced by ones of lower dimension. In particular, \mathcal{H} in the first exponential could be reduced to dimension 2^m , where (for $P = 0, \dots, N - 1$),

$$m = m_P = \text{integer part of } \left[\frac{(N - 1)L - P}{N} \right]. \tag{1.20}$$

Similarly, the second \mathcal{H} could be replaced by one of dimension $2^{m'}$, where $m' = m_Q$, and S_r by a 2^m by $2^{m'}$ matrix S_{PQ} . (p, q, S_{red}^r of paper II became P, Q, S_{PQ} in paper III and herein.)

We went on to conjecture that (1.18), (1.19) also applied to the superintegrable model, with fairly obvious generalizations of the definitions of the X, E, B matrices. We observed that this conjecture agreed with numerical tests performed to 60 digits of accuracy.

These calculations involved sets of m quantities $\theta_1, \dots, \theta_m$ defined by

$$\cos \theta_j = c_j = (1 + w_j)/(1 - w_j), \quad 0 \leq \theta_j < \pi, \tag{1.21}$$

where w_1, \dots, w_m are the zeros of the m th degree polynomial $\rho_P(w)$ given by

$$\rho_P(z^N) = z^{-P} \sum_{n=0}^{N-1} \omega^{nP} \left(\frac{1 - z^N}{1 - \omega^{-n}z} \right)^L, \tag{1.22}$$

taking $w = z^N$. Let $c = (1 + w)/(1 - w)$ and, for all complex numbers c ,

$$\mathcal{P}_P(c) = N^{-L}(c + 1)^m \rho_P(w). \tag{1.23}$$

Then $\mathcal{P}_P(c)$ is the polynomial with zeros c_1, \dots, c_m , i.e.

$$\mathcal{P}_P(c) = \prod_{j=1}^m (c - c_j). \tag{1.24}$$

Similarly, we can define m' quantities $\theta'_1, \dots, \theta'_{m'}$ and $c'_1, \dots, c'_{m'}$ by replacing P by Q in the above three equations. $\rho_P(w)$, $\mathcal{P}_P(c)$ here are those of paper III, which are $P(w)$, $\tilde{P}_P(c)$ of (II.2.17), (II.2.18) and (II.6.4).

In III we showed that both the S_r and S_{PQ} matrices satisfied various commutation relations with the Hamiltonians, in particular that S_{PQ} satisfied (III.3.39) and (III.3.40). We conjectured in (III.3.45) that the elements of S_{PQ} were simple ratios of products of trigonometric functions of θ_i . We suggested that this result applied for any values of θ_i and θ'_i , not necessarily those given by (1.21) and (1.22). Further, if we defined \mathcal{D}_{PQ} by (III.3.48), then it was also given as a determinant by (III.4.9) and (III.4.10), again for arbitrary θ_i, θ'_i .

Outstanding problems and progress

It therefore appears that there is indeed an algebraic route to calculate \mathcal{D}_{PQ} . However, there are still three outstanding problems to be overcome.

- (1) To prove that the elements of S_{PQ} are given by (III.3.45), and hence \mathcal{D}_{PQ} by (III.3.48).
- (2) To further prove that \mathcal{D}_{PQ} is given as a determinant by (III.4.9) or equivalently (III.4.10).
- (3) To calculate the determinant (III.4.9) in the limit $L, \alpha, \beta \rightarrow \infty$ so as to regain the known result (1.1). This has not previously been done directly even for the $N = 2$ Ising case: in paper I we calculated (1.1) from the expression for \mathcal{D}_{PQ} as the square root of an L by L determinant, using Szegő's theorem. This theorem was derived in response to the first (unpublished) derivation of the Ising model spontaneous magnetization by Onsager and Kaufman [27, 28]. It was later used by Montroll, Potts and Ward [21].

Progress has been made. We have proved that expression (III.3.45) for S_{PQ} satisfies the commutation relations (III.3.39)–(III.3.41). From numerical calculations for small N, L ($N, L \leq 6$), it appears that these relations (which are linear in the elements of S_{PQ} , with many more equations than unknowns) determine S_{PQ} uniquely. If so, then (III.3.45) and (III.3.48) have to be correct.

In paper III we defined y_i, y'_i to be the elements of the diagonal m - and m' -dimensional matrices $Y = X_P(\alpha)E_{PQ}$, $Y' = X_Q(\beta)E_{QP}$. Both expression (III.3.48) for \mathcal{D}_{PQ} as a $2^{m+m'}$ -dimensional sum, and expression (III.4.9) as an m -dimensional determinant are rational functions of c_i, c'_i, y_i, y'_i . One can take all these variables to be arbitrary and can verify that the denominators are identical. One can then prove that the numerators are also the same by a recursive method using the symmetries and the fact that if $c_m = c'_{m'}$, then each expression simplifies to one with m, m' replaced by $m - 1, m' - 1$.

In some ways the hardest of the three problems is to take the limit $L, \alpha, \beta \rightarrow \infty$. The determinant for \mathcal{D}_{PQ} is a hugely smaller calculation than the original 2^L -dimensional sums in (1.13a), (1.13b), but it is still ultimately infinite. We have succeeded in calculating the determinant in the limit $\alpha, \beta \rightarrow \infty$ as a simple product, for finite L, m, m' : the key trick is to note that when $\alpha, \beta \rightarrow \infty$ the matrix sum in (1.18) can be written as the product of two Cauchy-like matrices. Result (1.1) follows by then taking the $L \rightarrow \infty$ limit of the product. It is this calculation we report here. We hope to publish the work on the first two problems later.

2. The matrices

We shall need the definitions of the X and E diagonal matrices. From (II.3.16) and (II.7.4)

$$[X_P(\alpha)]_{i,j} = \frac{-k' \sin \theta_i \sinh(N\alpha\lambda_i)\delta_{i,j}}{\lambda_i \cosh(N\alpha\lambda_i) + (1 - k' \cos \theta_i) \sinh(N\alpha\lambda_i)}, \quad (2.1)$$

where

$$\lambda_i = (1 - 2k' \cos \theta_i + k'^2)^{1/2}. \quad (2.2)$$

The matrix $X_Q(\beta)$ is defined similarly, with $P, \alpha, \theta_i, \lambda_i$ replaced by $Q, \beta, \theta'_i, \lambda'_j$.

From (II.6.18) and (II.6.19), the matrix E_{PQ} is an m by m diagonal matrix with entries

$$[E_{PQ}]_{ij} = e(P, Q, i)\delta_{i,j}, \quad (2.3)$$

where, for $0 \leq P, Q < N$ and $P \neq Q$,

$$\begin{aligned} e(P, Q, i) &= \sin \theta_i && \text{if } P < Q \text{ and } m' = m - 1 \\ &= \tan(\theta_i/2) && \text{if } P < Q \text{ and } m' = m \\ &= 1/\sin \theta_i && \text{if } P > Q \text{ and } m' = m + 1 \\ &= \cot(\theta_i/2) && \text{if } P > Q \text{ and } m' = m. \end{aligned} \quad (2.4)$$

These equations cover all cases. The matrix E_{QP} is defined similarly, with P, Q interchanged, m, m' also interchanged and θ_i replaced by θ'_i .

B_{PQ} is an m by m' Cauchy-like matrix with elements

$$(B_{PQ})_{ij} = \frac{f_i f'_j}{c_i - c'_j}. \quad (2.5)$$

Given $c_1, \dots, c_m, c'_1, \dots, c'_{m'}$ with $|m - m'| \leq 1$, there is a unique way of choosing $f_1, \dots, f_m, f'_1, \dots, f'_{m'}$ so that B_{PQ} satisfies the orthogonality condition (1.19). The working is given in section 6 of II. We remark in section 4 of III that it is true for arbitrary c_i, c'_i . Let

$$\begin{aligned} a_i &= \prod_{j=1}^{m'} (c_i - c'_j), & a'_i &= \prod_{j=1}^m (c'_i - c_j), \\ b_i &= \prod_{j=1, j \neq i}^m (c_i - c_j), & b'_i &= \prod_{j=1, j \neq i}^{m'} (c'_i - c'_j); \end{aligned} \quad (2.6)$$

then results (II.6.8), (II.6.13), (II.6.16) can be written as

$$f_i^2 = \epsilon a_i / b_i, \quad f_i'^2 = -\epsilon a'_i / b'_i, \quad (2.7)$$

where $\epsilon = \pm 1$ is independent of i .

(For the particular values of c_i, c'_i given by (1.21) and (1.22), we observe numerically that f_i^2 and $f_i'^2$ are positive real if we choose $\epsilon = 1$ if $P < Q$, and $\epsilon = -1$ if $P > Q$.)

A quantity that we shall need is

$$\Delta_{m,m'}(c, c') = \frac{\prod_{1 \leq i < j \leq m} (c_i - c_j) \prod_{1 \leq i < j \leq m'} (c'_j - c'_i)}{\prod_{i=1}^m \prod_{j=1}^{m'} (c_i - c'_j)}. \quad (2.8)$$

3. The function $Z_P(\alpha)$

The partition function $Z_P(\alpha)$ is, from (II.3.16) and (II.5.38), or from (III.3.27) and (III.3.29),

$$Z_P(\alpha) = e^{-\mu_P \alpha} \prod_{i=1}^m \frac{\lambda_i \cosh(N\alpha\lambda_i) + (1 - k' \cos \theta_i) \sinh(N\alpha\lambda_i)}{\lambda_i}, \quad (3.1)$$

where

$$\mu_P = 2k'P + (1 + k')(mN - NL + L). \quad (3.2)$$

When α is large, $Z_P(\alpha)$ has the form $C e^{g\alpha}$, where C, g are independent of α and

$$g = g_P = -\mu_P + \sum_{i=1}^m \lambda_i. \quad (3.3)$$

Hence from (1.16), $Z_P(\alpha) \rightarrow C^{-1}$ as $\alpha \rightarrow \infty$.

Set

$$X_P = X_P(\infty), \quad x_i = (X_P)_{i,i}, \quad Z_P = Z_P(\infty) = C^{-1} \quad (3.4)$$

and let c, θ, λ be variables related to one another by

$$c = \cos \theta = (1 + k'^2 - \lambda^2)/2k'; \quad (3.5)$$

then c_i, θ_i, λ_i of (1.21), (2.2) are related in the same manner. Instead of viewing x_i, Z_P , etc as two-valued functions of c_i , we can regard them as single-valued rational functions of λ_i . Then

$$x_i^2 = \frac{k'^2 - (1 - \lambda_i)^2}{(1 + \lambda_i)^2 - k'^2}, \quad (3.6)$$

$$Z_P = \prod_{i=1}^m \frac{4\lambda_i}{(1 + \lambda_i)^2 - k'^2} = \prod_{i=1}^m (1 + x_i^2). \quad (3.7)$$

Analogous relations apply, with P, λ_i, x_i replaced by Q, λ'_i, x'_i , respectively.

Another function that we shall find useful is

$$\mathcal{R}(\lambda) = \frac{\prod_{i=1}^m (\lambda + \lambda_i)/2}{\prod_{j=1}^{m'} (\lambda + \lambda'_j)/2}, \quad (3.8)$$

together with the elementary identity

$$\left[\frac{\Delta_{m,m'}(\lambda^2, \lambda'^2)}{\Delta_{m,m'}(\lambda, \lambda')} \right]^2 = 2^{(m-m')^2} \prod_{i=1}^m \frac{\mathcal{R}(\lambda_i)}{2\lambda_i} \Big/ \prod_{j=1}^{m'} 2\lambda'_j \mathcal{R}(\lambda'_j). \quad (3.9)$$

4. Calculation of \mathcal{D}_{PQ}

For any m by m' matrix A , and m' by m matrix B , it is true that

$$\det(I_m + AB) = \det(I_{m'} + BA), \quad (4.1)$$

so from (1.18),

$$\begin{aligned} \mathcal{D}_{PQ}(\alpha, \beta) &= \det[I_{m'} - X_Q(\beta)E_{QP}B_{QP}X_P(\alpha)E_{PQ}B_{PQ}] \\ &= \mathcal{D}_{QP}(\beta, \alpha). \end{aligned} \quad (4.2)$$

This symmetry also follows directly from definitions (1.5)–(1.15), the fact that \mathcal{H} is Hermitian and $S_r^\dagger = S_{-r}$.

Without loss of generality, we can therefore restrict our attention to the case $P > Q$, when $m \leq m'$.

Then from (1.19) we can write I_m in (1.18) as $B_{PQ} B_{PQ}^T$. Remembering that $B_{QP} = -B_{PQ}^T$, we can then write (1.18) as

$$\mathcal{D}_{PQ}(\alpha, \beta) = \det [U B_{PQ}^T], \tag{4.3}$$

where

$$U = B_{PQ} + X_P(\alpha) E_{PQ} B_{PQ} X_Q(\beta) E_{QP}. \tag{4.4}$$

Define

$$y_i = [X_P(\alpha)]_{i,i} e(P, Q, i), \quad y'_j = [X_Q(\beta)]_{j,j} e(Q, P, j); \tag{4.5}$$

then from (2.5) the elements of U are

$$U_{ij} = \frac{f_i f'_j (1 + y_i y'_j)}{c_i - c'_j}. \tag{4.6}$$

In general we do not know how to calculate the determinant of such a matrix. However, if we take the limits $\alpha, \beta \rightarrow +\infty$ and express c_i, c'_j, y_i, y'_j as rational functions of λ_i, λ'_j , we find that a factor $\lambda_i + \lambda'_j$ cancels out of the RHS of (4.6). If $m' = m$, the result is a Cauchy-like matrix, and one can calculate the determinant of U .

Hereinafter we take the limit $\alpha, \beta \rightarrow +\infty$, so

$$y_i = x_i e(P, Q, i), \quad y'_j = x'_j e(Q, P, j), \tag{4.7}$$

where x_i, x'_j are given by (3.4). We write $\mathcal{D}_{PQ}(\infty, \infty)$ simply as \mathcal{D}_{PQ} . The integer L is still finite.

The case $P > Q, m = m'$

The simplest case is when $m = m'$ and all matrices are square. Then from (1.19) and (4.3),

$$\mathcal{D}_{PQ} = \det U / \det B_{PQ}. \tag{4.8}$$

Cauchy-like matrices. If A is the m by m matrix with entries

$$A_{ij} = \frac{1}{c_i - c'_j}, \tag{4.9}$$

then it is a Cauchy matrix and its determinant is $\Delta_{m,m}(c, c')$, using the definition (2.8) [29, equation (2.7)]. Any matrix with elements of the form (2.5) is said to be *Cauchy-like*, and has determinant

$$\det B_{PQ} = \Delta_{m,m}(c, c') \prod_{i=1}^m f_i f'_i \tag{4.10}$$

for all f_i, f'_i . We have in fact chosen f_i, f'_i so that B_{PQ} is orthogonal, so has determinant ± 1 . However, form (4.10) is convenient here as the f_i, f'_i products will cancel out of (4.8).

The determinant of U. From (4.8), we still have to calculate the determinant of U . Its elements are given by (4.6), so U is *not* in general Cauchy-like. However, in the limit $\alpha, \beta \rightarrow \infty$ we find that a common factor cancels from the numerator and denominator of (4.6), and U becomes Cauchy-like. We can then evaluate its determinant by parallelling (4.9) and (4.10).

The case $P > Q, m = m'$ is the fourth one listed in (2.4), so

$$(E_{PQ})_{ij} = \cot(\theta_i/2)\delta_{ij}, \quad (E_{QP})_{ij} = \tan(\theta'_i/2)\delta_{ij}. \tag{4.11}$$

From (2.1), taking the limit $\alpha \rightarrow +\infty$,

$$x_i = [X_P(\infty)]_{ii} = \frac{-2k' \sin \theta_i}{(1 + \lambda_i)^2 - k'^2}. \tag{4.12}$$

Noting that $\sin \theta_i \cot(\theta_i/2) = 1 + c_i = [(1 + k')^2 - \lambda_i^2]/2k'$, it follows from (4.7) that

$$y_i = -\frac{1 + k' - \lambda_i}{1 - k' + \lambda_i}. \tag{4.13}$$

(A factor $1 + k' + \lambda_i$ has been cancelled.)

The calculation of y'_j is similar, except that now we use $\sin \theta'_j \tan(\theta'_j/2) = 1 - c'_j = [\lambda_j^2 - (1 - k')^2]/2k'$ to obtain

$$y'_j = \frac{1 - k' - \lambda'_j}{1 + k' + \lambda'_j}. \tag{4.14}$$

Thus,

$$1 + y_i y'_j = \frac{2(\lambda_i + \lambda'_j)}{(1 - k' + \lambda_i)(1 + k' + \lambda'_j)}. \tag{4.15}$$

Also,

$$c_i - c'_j = (\lambda_j^2 - \lambda_i^2)/2k'. \tag{4.16}$$

We see that the factor $\lambda_i + \lambda'_j$ *cancels* out of (4.6), leaving

$$U_{ij} = \frac{-4k' f_i f'_j}{(1 - k' + \lambda_i)(1 + k' + \lambda'_j)(\lambda_i - \lambda'_j)} \tag{4.17}$$

so U is a Cauchy-like matrix, similar to B_{PQ} , but with the denominator $c_i - c'_j$ replaced by $\lambda_i - \lambda'_j$. Analogously to (4.10), its determinant is

$$\det U = \Delta_{m,m}(\lambda, \lambda') \prod_{i=1}^m \frac{-4k' f_i f'_i}{(1 - k' + \lambda_i)(1 + k' + \lambda'_i)}. \tag{4.18}$$

Also, from (4.16), we can write (2.5) as

$$(B_{PQ})_{ij} = \frac{-2k' f_i f'_j}{\lambda_i^2 - \lambda_j^2} \tag{4.19}$$

so $\det B_{PQ}$ is also equal to

$$\det B_{PQ} = \Delta_{m,m}(\lambda^2, \lambda'^2) \prod_{i=1}^m (-2k' f_i f'_i). \tag{4.20}$$

f_i, f'_i cancel out of the ratio (4.8), leaving

$$\mathcal{D}_{PQ} = \frac{\Delta_{m,m}(\lambda, \lambda')}{\Delta_{m,m}(\lambda^2, \lambda'^2)} \prod_{i=1}^m \frac{2}{(1 - k' + \lambda_i)(1 + k' + \lambda'_i)}. \tag{4.21}$$

From (1.17), (3.7), (3.8) and (3.9) it follows that

$$(\mathcal{M}_r^{(2)})^2 = \frac{\mathcal{R}(1 + k') \prod_{j=1}^{m'} \mathcal{R}(\lambda'_j)}{\mathcal{R}(1 - k') \prod_{i=1}^m \mathcal{R}(\lambda_i)}. \tag{4.22}$$

All cases

When $P > Q$ and $m' = m + 1$, we can still write \mathcal{D}_{PQ} as in (4.3). However, U and B_{PQ} are no longer square matrices, so we can no longer simply take products or ratios of determinants, as in (4.8). Even so, we can still calculate \mathcal{D}_{PQ} (and hence $\mathcal{M}_r^{(2)}$) by adding a row to B_{PQ} and U to make them square Cauchy-like matrices. The matrix UB_{PQ}^T in (4.3) is then m' by m' , but is of an upper-block-triangular form. The top-left block is the original m by m matrix UB_{PQ}^T , while the lower-right block is the 1 by 1 unit matrix. Hence, the determinant (4.3) is unchanged and can be evaluated as a product of the two m' by m' determinants.

We do this in the appendix. Result (A.18) is the same as (4.22), except that the factor $\mathcal{R}(1 + k')$ is inverted.

From (1.17) and (4.2), $\mathcal{M}_r^{(2)}$ is unchanged by interchanging P with Q , m with m' and λ_i with λ'_j . From (3.8), this inverts the function \mathcal{R} . We can use this symmetry to calculate $\mathcal{M}_r^{(2)}$ in the other two cases.

For the four cases (2.4), define the factor \mathcal{G} by

$$\begin{aligned} \mathcal{G} &= \mathcal{R}(1 - k')\mathcal{R}(1 + k') && \text{if } P < Q \text{ and } m' = m - 1 \\ &= \mathcal{R}(1 - k')/\mathcal{R}(1 + k') && \text{if } P < Q \text{ and } m' = m \\ &= 1/[\mathcal{R}(1 - k')\mathcal{R}(1 + k')] && \text{if } P > Q \text{ and } m' = m + 1 \\ &= \mathcal{R}(1 + k')/\mathcal{R}(1 - k') && \text{if } P > Q \text{ and } m' = m; \end{aligned} \tag{4.23}$$

then we find that

$$(\mathcal{M}_r^{(2)})^2 = \mathcal{G} \prod_{j=1}^{m'} \mathcal{R}(\lambda'_j) \bigg/ \prod_{i=1}^m \mathcal{R}(\lambda_i) \tag{4.24}$$

for all four cases.

5. The limit $L \rightarrow \infty$

Result (4.24) is exact for finite L . The last step in the calculation is to let $L \rightarrow \infty$. For this we shall need for the first time herein the particular definition (1.22) and (1.23) of the polynomial (1.24).

Let c, λ be two variables related as are c_i, λ_i in (3.5). Noting that $\lambda_i^2 - \lambda^2 = 2k'(c - c_i)$, it follows from (3.8) and (1.24) that

$$\mathcal{R}(\lambda)\mathcal{R}(-\lambda) = (k'/2)^{m-m'} \mathcal{P}_P(c)/\mathcal{P}_Q(c). \tag{5.1}$$

Using (1.22), (1.23), this can be written as

$$\mathcal{R}(\lambda)\mathcal{R}(-\lambda) = [k'(c + 1)/2]^{m-m'} z^{Q-P} \mathcal{W}(\lambda), \tag{5.2}$$

where

$$\mathcal{W}(\lambda) = \mathcal{W}_P(\lambda)/\mathcal{W}_Q(\lambda), \tag{5.3}$$

$$\mathcal{W}_P(\lambda) = 1 + \sum_{n=1}^{N-1} \omega^{n(P+L)} \left(\frac{1 - z}{\omega^n - z} \right)^L. \tag{5.4}$$

We have cancelled the $n = 0$ term in (1.22) from the ratio (5.3).

These z, w, c are related to λ by relations (1.22), (1.23), (3.5). In particular,

$$z^N = w = \frac{c - 1}{c + 1} = \frac{(1 - k')^2 - \lambda^2}{(1 + k')^2 - \lambda^2}. \tag{5.5}$$

When $\lambda^2 < (1 - k')^2$ we choose z to be positive real. Thus, w, c are rational functions of λ . We see that z is multi-valued, but we can choose it to be analytic by cutting the complex λ -plane from $1 - k'$ to $1 + k'$, and from $-1 - k'$ to $-1 + k'$. It is then analytic on the imaginary axis and at infinity.

The function $\mathcal{W}(\lambda)$ is even and rational, its poles and zeros being symmetrical about the imaginary axis, with none on the axis. Also, $z \rightarrow 1$ and $\mathcal{W}(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$, so $\log \mathcal{W}(\lambda)$ is analytic in a vertical strip containing the imaginary axis, and tends to zero at infinity as $1/\lambda^2$.

We can therefore perform a Wiener–Hopf factorization [30] of $\mathcal{W}(\lambda)$:

$$\mathcal{W}(\lambda) = \mathcal{W}_+(\lambda)\mathcal{W}_-(\lambda), \tag{5.6}$$

where, for $\Re(\lambda) \geq 0$,

$$\log \mathcal{W}_+(\lambda) = -\frac{1}{2\pi i} \int_{-i\infty-\epsilon}^{i\infty-\epsilon} \frac{\log \mathcal{W}(\lambda')}{\lambda' - \lambda} d\lambda', \tag{5.7}$$

the integration being up a vertical line just to the left of the imaginary axis. This $\mathcal{W}_+(\lambda)$ is analytic and non-zero in the right-half complex λ -plane and on the imaginary axis, and tends to 1 as $\lambda \rightarrow \infty$. The function $\mathcal{W}_-(\lambda)$ is defined similarly, but with $\Re(\lambda) \leq 0$ and the sign of the RHS reversed: it is analytic and non-zero in the LHP and on the imaginary axis.

From (3.8), the function $\mathcal{R}(\lambda)$ is analytic in the RHP and is proportional to $(\lambda/2)^{m-m'}$ when $\lambda \rightarrow \infty$. It follows from (5.5) and

$$k'(c+1)/2 = \frac{(1+k')^2 - \lambda^2}{4}$$

that

$$\mathcal{R}(\lambda) = \left(\frac{1+\lambda+k'}{2}\right)^{m-m'} \left(\frac{1-k'+\lambda}{1+k'+\lambda}\right)^{(Q-P)/N} \mathcal{W}_+(\lambda). \tag{5.8}$$

This is an exact result, true for finite L .

Let

$$t = \left(\frac{1-k'}{1+k'}\right)^{2/N};$$

then when λ lies on the imaginary axis, z lies on the positive real axis, between t and 1. Hence, for $0 < n < N$,

$$\left|\frac{1-z}{\omega^n - z}\right| \leq \left|\frac{1-t}{\omega - t}\right| < 1. \tag{5.9}$$

It follows from (5.3), (5.4) that $\mathcal{W}(\lambda)$ tends uniformly to 1 as $L \rightarrow \infty$. For sufficiently small ϵ , this must also be true for λ on the integration line in (5.7). Hence

$$\mathcal{W}_+(\lambda) \rightarrow 1 \quad \text{as } L \rightarrow \infty, \quad \text{for } \Re(\lambda) \geq 0, \tag{5.10}$$

so $\mathcal{R}(\lambda)$ is then given by (5.8) with $\mathcal{W}_+(\lambda) = 1$.

It follows that

$$\prod_{j=1}^{m'} \mathcal{R}(\lambda'_j) \Big/ \prod_{i=1}^m \mathcal{R}(\lambda_i) = \mathcal{R}(1+k')^{m'-m} \left[\frac{\mathcal{R}(1+k')}{\mathcal{R}(1-k')}\right]^{(Q-P)/N}, \tag{5.11}$$

$$\mathcal{R}(1+k') = (1+k')^{m-m'+(P-Q)/N}, \quad \mathcal{R}(1-k') = (1-k')^{(Q-P)/N}. \tag{5.12}$$

Using $r = Q - P$, modulo N , we obtain

$$(\mathcal{M}_r^{(2)})^2 = (1-k'^2)^{r(N-r)/N^2} \tag{5.13}$$

for all four cases, in agreement with (1.1).

Asymptotic degeneracy

We are now in a position to confirm the remarks we made before (1.14). Expanding (3.8) and (5.8) to first order in $1/\lambda$ for λ large and equating the coefficients, using (3.2) and (3.3), we obtain

$$g_P = g_Q.$$

We noted after (3.3) that when α is large, then $Z_P(\alpha)$ is proportional to $\exp(g_P\alpha)$, so $Z_P(\alpha)/Z_Q(\alpha)$ must tend to a limit as $\alpha \rightarrow \infty$. From (1.16) and (3.4), this is Z_Q/Z_P , so from (3.7) and (3.8),

$$\lim_{\alpha, L \rightarrow \infty} \frac{Z_P(\alpha)}{Z_Q(\alpha)} = \frac{\mathcal{R}(1-k')\mathcal{R}(1+k')}{2^{m-m'}\mathcal{R}(0)} = 1, \tag{5.14}$$

using (5.8) with $\mathcal{W}_+(\lambda) = 1$ to evaluate the middle expression.

Hence in the limit $\alpha, \beta, L \rightarrow \infty$, $Z_P(\alpha)$ is the same for all P . We have already shown in (5.13) that $\mathcal{M}_r^{(2)}$ depends on P and Q only via the difference r . From (1.15)–(1.17), the same must be true of $W_{PQ}(\alpha, \beta)$. This justifies our replacing the sums in (1.12) (for $b = 0$) by single terms.

6. Summary

The magnetization \mathcal{M}_r of the chiral Potts model can be expressed as the expectation value of ω^{ra} , where a is a spin inside a cylindrical lattice with the spins at the top and bottom boundaries fixed to zero. This was calculated in [3, 4] analytically by generalizing \mathcal{M}_r , showing that it satisfied functional relations in the large-lattice limit, and then solving those relations. This was quite a different method from the algebraic techniques originally used for the Ising model.

To obtain \mathcal{M}_r for the general solvable model it is sufficient to obtain it for the superintegrable case, and this case has many resemblances to the Ising model. This naturally leads to the question whether there is an algebraic way of calculating the magnetization for the superintegrable chiral Potts model.

We looked at this problem in three previous papers [23–25], which we refer to as I, II, III. In I we revisited the Ising model (the $N = 2$ case of the superintegrable model) and showed that \mathcal{M}_r was proportional to the determinant (I.7.7) and (I.7.9).

In II we first showed in (II.5.37) that \mathcal{M}_r was proportional to the weighted sum of the elements of a 2^m by $2^{m'}$ matrix S_{red}^r . Generalizing our calculation in I, we then conjectured that it was proportional to the m by m' determinant $\mathcal{D}_{PQ}(\alpha, \beta)$ of (1.18) herein.

In III we further showed that the matrix $S_{PQ} = S_{\text{red}}^r$ satisfied a number of commutation relations, in particular the two relations (2.21), (2.22) therein, and further conjectured that the elements of S_{PQ} had the particular simple product form (III.3.45).

We have since proved that this form for S_{PQ} both satisfies the two commutation relations and implies the determinantal result. We hope to publish the working soon. What we have not done, and would be needed to complete the proof, is to show that the commutation relations (III.3.39) and (III.3.40) (plus the simple normalization property (III.3.41)) define S_{PQ} uniquely. Calculations for small lattices suggest that this is so.

These calculations are all for finite sums and determinants of finite matrices. They involve parameters α, β , which can be regarded as a measure of the number of rows below and above the selected spin a in the lattice. They also involve the lattice width L . Ultimately we want to take the limits $\alpha, \beta, L \rightarrow \infty$.

To complete this algebraic calculation of \mathcal{M}_r , we need to evaluate the determinant $\mathcal{D}_{PQ}(\alpha, \beta)$. We do not know how to do this for finite α, β , but we show in section 4 and the

appendix herein that in the limit when α and β are both infinite, $\mathcal{D}_{PQ}(\alpha, \beta)$ can be written as the determinant of the product of two square Cauchy-like matrices. We can therefore evaluate the determinant as a product of terms, the number of terms being quadratic in m, m' . Finally in section 5 we take the limit $L \rightarrow \infty$, showing that the needed function $\mathcal{R}(\lambda)$ then has a simple form for $\Re(\lambda) \geq 0$. We of course regain result (1.1) of [3, 4], which had been conjectured by Albertini *et al* in 1989 [1].

For the $N = 2$ Ising case, this appears to be a new way of calculating the needed determinant. Yang [20] did so by calculating the eigenvalues in the large- L limit. Montroll *et al* [21], and presumably Onsager and Kaufman in 1949 [27, 28], did so by expressing the result in terms of a Toeplitz determinant and then using Szegő's theorem [31].

7. Additions

Since posting this paper on the Los Alamos archive, there has been considerable further progress on the first two problems listed above in the Introduction. Iorgov *et al* [32] have proved that S_{PQ} is indeed given by (III.4.9), and therefore D_{PQ} by (III.3.48). They did this by showing that (III.3.45) satisfied the commutation relations (III.3.39)–(III.3.41), and were able to prove that these relations have a unique solution. Further, they went on to calculate D_{PQ} in the limit $\alpha, \beta \rightarrow \infty$ directly, using the expression (III.3.48) as a sum over matrix elements, rather than the determinantal form (III.4.9) or (III.4.10).

The author has also posted a paper on the archive [33] giving the proof of the equivalence of the sum and determinantal forms of D_{PQ} that is outlined in the penultimate paragraph of the introduction above. It is remarked therein that the motivation for this work was the similarity between the Ising and superintegrable chiral Potts models, in particular to find a derivation of the determinant form of D_{PQ} . This determinantal form was conjectured in II by generalizing the Ising model result as expressed in I.

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Appendix

Here we consider the case when $P > Q$ and $m' = m + 1$, so the matrices B_{PQ}, U are not square. Then from (2.4), $e(P, Q, i) = 1/\sin \theta_i$ and $e(Q, P, j) = \sin \theta'_j$. Using (4.5), (4.12), we obtain

$$y_i = \frac{-2k'}{(1 + \lambda_i)^2 - k'^2}, \quad y'_j = \frac{(1 - \lambda'_j)^2 - k'^2}{2k'}. \tag{A.1}$$

Hence

$$1 + y_i y'_j = \frac{(\lambda_i + \lambda'_j)(2 + \lambda_i - \lambda'_j)}{(1 + \lambda_i)^2 - k'^2}. \tag{A.2}$$

Using (4.16), we see that the factor $\lambda_i + \lambda'_j$ again cancels in (4.6), leaving

$$U = U^{(1)} + U^{(2)}, \tag{A.3}$$

where $U^{(1)}, U^{(2)}$ are matrices with elements

$$U_{ij}^{(1)} = \frac{-4k' f_i f_j'}{(\lambda_i - \lambda_j')[(1 + \lambda_i)^2 - k'^2]}, \tag{A.4}$$

$$U_{ij}^{(2)} = \frac{-2k' f_i f_j'}{(1 + \lambda_i)^2 - k'^2}, \tag{A.5}$$

respectively. Thus,

$$U^{(2)} = \xi \eta^T, \tag{A.6}$$

where ξ, η are vectors and $(\eta)_j = f_j'$.

Consider the vector $B_{PQ} \eta$: from (2.5) it has entries

$$(B_{PQ} \eta)_i = f_i \mathcal{F}'(c_i),$$

where

$$\mathcal{F}'(c) = \sum_{j=1}^{m'} \frac{f_j'^2}{c - c_j'}.$$

Remembering that $P > Q$, this is precisely the sum considered in (II.6.6), but with p, q therein replaced by Q, P , so (also using (II.6.14))

$$\mathcal{F}'(c) = \gamma' + \mathcal{P}_P(c)/\mathcal{P}_Q(c), \tag{A.7}$$

where γ' is independent of c . Taking the limit $c \rightarrow \infty$, we obtain $\gamma' = 0$. It follows that $\mathcal{F}'(c)$ vanishes when $c = c_i$ ($i = 1, \dots, m$), so

$$B_{PQ} \eta = 0. \tag{A.8}$$

Substituting (A.3) into (4.3), the $U^{(2)}$ term is zero, leaving

$$\mathcal{D}_{PQ} = \det [U^{(1)} B_{PQ}^T]. \tag{A.9}$$

We have reduced the problem to one of calculating the determinant of a product of two Cauchy-like matrices, but unfortunately they are not square. The solution to this problem is actually suggested by (A.8). Define an m' by m' matrix

$$\mathcal{B} = \begin{pmatrix} B_{PQ} \\ \eta^T \end{pmatrix}, \tag{A.10}$$

(dropping the suffixes P, Q). Then, using (1.19), all the m' rows of \mathcal{B} are mutually orthogonal. Multiplying (A.7) by c and taking the limit $c \rightarrow \infty$, we obtain

$$\eta^T \eta = \sum_{j=1}^{m'} f_j'^2 = 1, \tag{A.11}$$

so together with (A.8) it follows that $\mathcal{B}\mathcal{B}^T = I_{m'}$, i.e. \mathcal{B} is a square orthogonal matrix.

We also extend $U^{(1)}$ by adding the row η^T to form the square matrix

$$\mathcal{U} = \begin{pmatrix} U^{(1)} \\ \eta^T \end{pmatrix}. \tag{A.12}$$

Then, using (A.8) and (A.11),

$$\mathcal{U}\mathcal{B}^T = \begin{pmatrix} U^{(1)} B_{PQ}^T & U^{(1)} \eta \\ \eta^T B_{PQ}^T & \eta^T \eta \end{pmatrix} = \begin{pmatrix} U^{(1)} B_{PQ}^T & U^{(1)} \eta \\ \mathbf{0} & 1 \end{pmatrix}. \tag{A.13}$$

We see that UB^T is an upper-right block triangular matrix and $\det UB^T = \det U^{(1)} B_{PQ}^T$. From (A.9), using the orthogonality of \mathcal{B} ,

$$\mathcal{D}_{PQ} = \det U / \det \mathcal{B}. \tag{A.14}$$

The square matrices \mathcal{B} and U are Cauchy-like. All the elements \mathcal{B}_{ij} of \mathcal{B} are given by the RHS of (4.19), provided we take $f_{m+1} = -\lambda_{m+1}^2 / 2k'$ and then let $\lambda_{m+1} \rightarrow \infty$. Using the general formula (4.10) for an m' by m' determinant, and then taking this limit, we obtain

$$\det \mathcal{B} = \Delta_{m,m'}(\lambda^2, \lambda'^2) \prod_{i=1}^m (2k' f_i) \prod_{j=1}^{m'} f'_j, \tag{A.15}$$

where $\Delta_{m,m'}(c, c')$ is defined by (2.8). (Similar to section 4, we know from the orthogonality of \mathcal{B} that its determinant is ± 1 , but this form is convenient here because the f_i, f'_j products will cancel out of (A.14).)

Similarly, all the elements \mathcal{U}_{ij} of U are given by the RHS of (A.4), except that now we take $f_{m+1} = -\lambda_{m+1}^3 / 4k'$ before letting $\lambda_{m+1} \rightarrow \infty$. Again using the general formula (4.10), we find that

$$\det U = \Delta_{m,m'}(\lambda, \lambda') \prod_{i=1}^m \frac{4k' f_i}{(1 + \lambda_i)^2 - k'^2} \prod_{j=1}^{m'} f'_j. \tag{A.16}$$

Hence from (A.14),

$$\mathcal{D}_{PQ} = \frac{\Delta_{m,m'}(\lambda, \lambda')}{\Delta_{m,m'}(\lambda^2, \lambda'^2)} \prod_{i=1}^m \frac{2}{(1 + \lambda_i)^2 - k'^2}. \tag{A.17}$$

Using (1.17), (3.7), (3.8) and (3.9), we now obtain

$$(\mathcal{M}_r^{(2)})^2 = \frac{\prod_{j=1}^{m'} \mathcal{R}(\lambda'_j)}{\mathcal{R}(1 + k') \mathcal{R}(1 - k') \prod_{i=1}^m \mathcal{R}(\lambda_i)}. \tag{A.18}$$

This is the result quoted at the end of section 4.

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